FREE LEFT DISTRIBUTIVE IDEMPOTENT SEMIGROUPS

Rushadije Ramani Halili(1) and Merime Mustafi (2)

Department of Mathematics

State University of Tetova

Avenue “Blagoja Toska” Bllok 82-A1/6

Tetova 1200, Fyrom Makedonia

rushadije.ramani@unite.edu.mk , **merime_mustafi@yahoo.com**

Abstract. The paper [1] represents construction of free left distributive semigroups - by J. Jezek and T. Kepka. In that paper Jezek and Kepka has also shown, that proceeding similarly, one can construct free left distributive idempotent semigroups.

In this paper it is presented the construction of free left distributive idempotent semigroups.

Key words: Free distributive semigroups, idempotent, permutation.

0.INTRODUCTION

A semigroup S is called left(right) distributive semigroup if , $xyz = xyxz$ ($yzx = yxzx$)where x, y, z belong to S.

Semigroup S is called distributive if it is both left and right distributive.

A semigroup is called

- medial if it satisfies the equation $xuvy = xvuy$;

-left (right) semimedial if it satisfies the identity $x^2yz = xyxz(zyx^2 = zxyx)$;

-semimedial if it is both left and right semimedial;

-middle semigroup if it satisfies the identity $xyzx = xzyx$.

ISSN 2229-5518

The element $x \in S$ is an idempotent if $x^2 = x$. Also we define the set of idempotents in S to be $Id(S) = \{x \in S \mid xx = x\}$

Let A be a finite alphabet, $F[A]$ denote the set of nonempty words $a_1a_2....a_n$ over A. The binary operation "$\cdot$" an $F[A]$ is defined

$$(a_1a_2...a_n)(b_1b_2...b_n) = a_1a_2...a_nb_1b_2...b_n \qquad ...(*)$$

The set $F[A]$ with a binary multiplicative operations defined by $(*)$ is called free semigroups over A.

1. EXAMPLES

Some examples of left distributive idempotent semigroups

	a	b
a	a	a
b	a	b

	a	b
a	a	b
b	a	b

	a	b
a	a	a
b	b	b

	a	b	c
a	a	a	a

b	a	b	b
c	a	b	c

	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

	a	b	c
a	b	a	a
b	a	b	a
c	a	b	c

2. CONSTRUCTION OF FREE LEFT DISTRIBUTIVE IDEMPOTENT SEMIGROUPS

Let X be a nonempty set. Denote by $\mathbf{F}$ the free semigroup over X .

Denote by F the union of the following two pairwise disjoint subsets A, B of **F:**

$A = \{x_1 x_2 ... x_n ;\quad x_1, x_2, ..., x_n \in X, n \geq 1$ pairwise different $\}$

$B = \{x_1 x_2 ... x_n x_k ;\quad x_1, x_2, ..., x_n \in X ,\ n \geq 2, 1 \leq k \leq n$ pairwise different $\}$ **.**

For every element u of **F**, expressed as $u = x_1 ... x_n$ where $n \geq 1; x_i \in X, 1 \leq i \leq n$ and $x_1 \neq x_2 \neq x_3 \neq ... \neq x_n$, we define an element $f(u)$ of F as follows:

(i) If $n = 1$, let $f(u) = x_1$

(ii) If $n = 2$, let $f(u) = x_1 x_2$

(iii) If $n \geq 3$ and $x_n \notin \{x_1, x_2, ..., x_{n-1}\}$ let $f(u) = x_1 y_1 ... y_m x_n$ and (by induction on i) y_i is the fist member of $x_1, ..., x_{n-1}$ not contained in $\{x_1, y_1, ..., y_{i-1}\}$.

(iv) If $n \geq 3$ and $x_n \in \{x_1, ..., x_{n-2}\}$, let $f(u) = x_1 y_1 ... y_n x_n$ and (by induction on i) y_i is the first member of $x_1, ..., x_{n-1}$ not contained in $\{x_1, y_1, ..., y_{i-1}\}$.

It is easy to see that $f(u) \in F$ in any case. Also, it is easy to see that $f(u) = u$ for $u \in F$. Let us define a binary operation $*$ on F in this way: $u * v = f(u)$ for any $u, v \in F$. We are going to prove that $F(*)$ is a free left distributive idempotent semigroup over X.

Proposition 1: Let $u, v \in F$ and $u \neq v$. Then there is an left distributive idempotent semigroup not satisfying $u = v$.

Proof: Suppose that $u = v$ is satisfied in all left distributive idempotent semigroup. Since every left zero semigroup is left distributive the words u, v hare the same first letters. Similarly right zero semi group is left distributive and hence u, v have the same last letters. Furthermore, every semilattice is distributive and we conclude that the set of letters accurring in u coincides with the set of letters occurring in v. Now, we distinguish the following cases.

Case1: $u = x_1 x_2 \ldots x_{n-1} x_n \in A$ and $v = x_{p(1)} x_{p(2)} \ldots x_{p(n-1)} x_{p(n)} \in A$ for a permutation p of $\{1, \ldots, n\}$ with $p(1) = 1$ and $p(n) = n$. If $n \geq 4$, then every left distributive idempotent semigroup satisfying $u = v$ is medial.

Hewever, semigroup in example

	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

is a non- medial left distributive idempotent semigroup. Consequently $n = 3$.

Case 2: $u = x_1 x_2 \ldots x_{n-1} x_n \in A$ and $v = x_{p(1)} x_{p(2)} \ldots x_{p(n-1)} x_{p(n)} x_{p(k)} \in B$ for a permutation p of $\{1, \ldots, n\}$ with $p(1) = 1$ and $p(k) = n$. One can easily check that every left distributive idempotent semigroup satisfying $u = v$ is distributive. However, semigroup in example is not distributive, a contradiction.

	a	b	c
a	a	a	a
b	b	b	b

ISSN 2229-5518

c	a	b	c

Case 3: $u = x_1 x_2 \ldots x_{n-1} x_k \in B$ and $v = x_{p(1)} x_{p(2)} \ldots x_{p(n-1)} x_{p(n)} x_{p(l)} \in B$ for a permutation p of $\{1,\ldots,n\}$ with $p(1) = 1$ and $p(l) = k$.

Since the semigroup

	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

is not middle semimedial, we have $p(2) = 2, \ldots, p(n) = n$.

However, the left distributive idempotent semigroup in example does not satisfy $axa = a$

	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

Thus, $u = v$, a contradiction. #

Theorem: For a nonempty set X, the grupoid $F(*)$ constructed in 2 is a free left distributive idempotent semigroup over X.

Proof: Denote by □ the set of the ordered pairs (u, v) of elements of **F** such that the equation $u = v$ is satisfied in all left distributive idempotent semigroups. So, □ is a congruence of **F** and **F** / □ is a free left distributive idempotent semigroup over X .

It is obvious that $f(u) = u$ for any **F** so that (by proposition 1) u □ v if and only if $f(u) = f(v)$ for any $u, v \in$ **F** and □ is just the kernel of f .

Now, f is a homomorphism of **F** onto $F(*)$: if $u, v \in$ **F**, then both $f(uv)$ and $f(u) * f(v)$ belong to F and are congruent modulo □ with uv .

The result follows from the homomorphism theorem.

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